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Synchronization in chaotic Jerk dynamical systems

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Abstract : In this communication we have discussed the synchronization of chaos in *Jerk dynamical system*. We have in particular applied three techniques for synchronization of chaos namely : (i) complete replacement (CR), (ii) feed back and (iii) adaptive control techniques (ACA). The stability of the synchronization in all the aforesaid methods have been discussed in detail.

Keywords : Synchronization, chaos, jerk equation.

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1. Introduction

Synchronization of chaos has attracted considerable interest among researchers during last decade due to its potential application in communication [1–3]. There is no possibility for two chaotic systems to behave in a synchronized way. One of the main feature associated with the definition of chaotic motion is its sensitive dependence on initial conditions. Therefore synchronization is not feasible in chaotic systems because it is not possible in real physical systems to reproduce exactly identical initial conditions and system parameters of two similar systems. We can build nearly identical systems but there is always a technology mismatch and noise, impeding the exact reproduction of the parameters and initial conditions. Thus an infinitesimal difference either in any one of the system parameters or in initial conditions of the trajectories will eventually result in the divergence of the trajectories.

Recently a number of algorithms have been suggested in literature [4–8] for synchronization of chaos and have been successfully applied theoretically and experimentally to various chaotic systems like Rössler, Lorenz, Logistic equation, Chua Circuit, double scroll oscillator and laser systems *etc.*

In this paper we discuss the synchronization of newly developed chaotic systems [9,10] involving *Jerk equation*. *Jerk equation* is an ordinary third order differential equation in one real scalar dynamical variable. The functional form of the jerk equation is $\dddot{x} = j(\ddot{x},\dot{x},x)$, here $\dddot{x}$ is rate of change of acceleration and is called jerk. We will consider here following form of *Jerk equation*

$$\dddot{x} + A\ddot{x} + \dot{x} = G(x), \tag{1}$$

where $G(x)$ is a linear piecewise function. We will consider $G(x) = |x| - 2$ throughout this paper. Other forms of $G(x)$ that exhibit chaotic motion are given in the reference [9]. Under certain restrictions jerky dynamics can be interpreted as the direct extension of one-dimensional Newtonian dynamics $(\ddot{x} = F)$ of a particle of unit mass where the force F is addition of (i) an instantaneous force that involves frictional force $(-A\dot{x})$ and the motion in quadratic potential and (ii) a non local force $F_{nl} = \int_0^t G(t)dt$ or memory term that integrates over positional history of motion. We can simply say that *Jerk equation* represents damped harmonic oscillator driven by a non linear memory term involving integral of $G(x)$ *i.e.*

$$\ddot{x} + A\dot{x} + x = \int^t G(t)dt. \tag{2}$$

We would also like to mention here that the six dimensional solar wind driven magnetosphere ionosphere (WINDMI) model [11] can be reduced to eq. (1) by using reasonable approximations and only difference would be in the form of $G(x)$. Sprott [9] has shown that equation (1) shows bifurcation route to chaos in the parameter range $0.55 < A < 0.8$. For $A = 0.6$ chaotic attractor is shown in Figure 1a and maximum Lyapunov spectrum for the range $0.55 < A < 0.8$ in step of 0.001 is shown in Figure 1b. We present in Section 2, outline of the synchronization methods, which we have used for our present work. In Section 3, the results and the importance of the present calculation for the *Jerk dynamical system* are discussed.

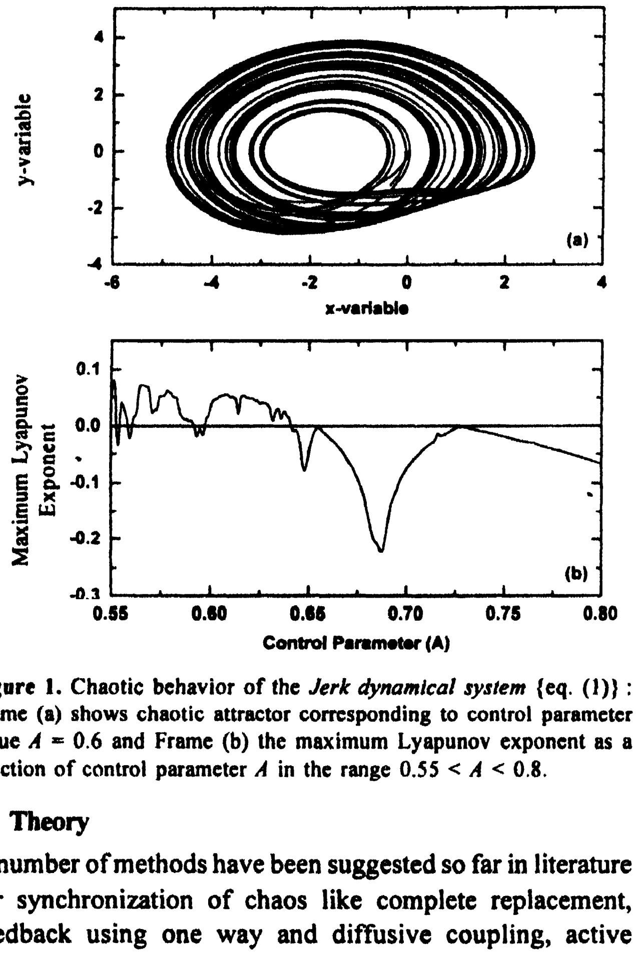

Figure 1. Chaotic behavior of the *Jerk dynamical system* {eq. (1)} : Frame (a) shows chaotic attractor corresponding to control parameter value $A = 0.6$ and Frame (b) the maximum Lyapunov exponent as a function of control parameter A in the range $0.55 < A < 0.8$.

2. Theory

A number of methods have been suggested so far in literature for synchronization of chaos like complete replacement, feedback using one way and diffusive coupling, active passive decomposition, adaptive control algorithm, addition of external noise *etc*. We refer the reader for different methods for synchronization of chaos to the following references [3-8]. We discuss in this section in brief the basic concept and stability criterion of complete replacement [4], feedback [5] and adaptive control techniques [6] for synchronization.

2.1. Complete Replacement (CR) technique :

This method was suggested by Pecora and Carroll [4]. We discuss the method in brief here by considering a nonlinear dynamcial system having three variables x, y and z. The evolution of the system is given in terms of flow functions f, g and h as :

$$\dot{x} = f(x,y,z);\ \dot{y} = g(x,y,z);\ \dot{z} = h(x,y,z). \tag{3}$$

In case of complete replacement technique we start with two identical chaotic systems having same system parameters (but not exactly same initial conditions) and completely replace one of the variables in one system by its counterpart in the other system. We consider drive and response systems which are given as :

$$\begin{aligned} \dot{x}_1 &= f(x_1,y_1,z_1), & & x_2 = x_1, \\ \dot{y}_1 &= g(x_1,y_1,z_1), & \text{and}\quad & \dot{y}_2 = g(x_2,y_2,z_2), \\ \dot{z}_1 &= h(x_1,y_1 z_1) & & \dot{z}_2 = h(x_2,y_2,z_2). \end{aligned} \tag{4}$$

We solve simultaneously drive and response systems given by eq. (4), for $t \to \infty$ leads to $|y_1 - y_2|$ and $|z_1 - z_2| \to 0$ and we end up the constraint $y_1 = y_2$ and $z_1 = z_2$. For understanding the stability of the above synchronization we perform a transformation to a new set of coordinates

$$y_\perp = y - y_2 \quad \text{and} \quad z_\perp = z_1 - z_2$$
$$y_\parallel = y_1 + y_2 \quad \text{and} \quad z_\parallel = z_1 + z_2$$

in which three coordinates (x_1, $y_\parallel$, $z_\parallel$) are on the synchronization manifold and $(y_\perp, z_\perp)$ on transverse manifold. For a stable synchronization we need $z_\perp \to 0$ and $y_\perp \to 0$ as $t \to \infty$. Thus the point (0,0) on the transverse manifold must be a fixed point. We obtain on applying the transformations in equation (4) the relation,

$$\begin{pmatrix} \dot{y}_\perp \\ \dot{z}_\perp \end{pmatrix} = DF.\begin{pmatrix} y_\perp \\ z_\perp \end{pmatrix}, \tag{5}$$

here DF is Jacobian $\dfrac{\partial F_i}{\partial x_i}$. The solutions of the eq. (5) convey us about the divergence and convergence of two initially nearby trajectories, so we treat response subsystem (y_2, z_2) as a separate system driven by x_1 and calculate the Lyapunov exponents. These exponents depend on x_1 and hence called conditional Lyapunov exponents (CLE's). If all the solutions are negative then both trajectories converge and synchronization is possible, hence negativity of all the CLE's is a necessary condition for stable synchronization.

2.2. Feedback technique :

Feedback method has been used by many researchers (Singer *et al* [12], Chen and Dong [13], Pyragas [14]) for control of chaos. We can achieve synchronization by using combination of Pecora and Carroll technique [4] and feedback method for controlling chaos [5]. We choose a drive variable from the drive system and feedback control is applied to the response system. The feedback control is directly proportional to the difference of a dynamical variable from drive and response systems. We consider the three-dimensional dynamical system as given in eq. (3) and the corresponding drive and response systems are given as :

$$\dot{x}_1 = f(x_1,y_1,z_1), \quad \dot{x}_2 = f(x_2,y_2,z_2) - c(x_1 - x_2),$$
$$\dot{y}_1 = g(x_1,y_1,z_1), \quad \text{and} \quad \dot{y}_2 = g(x_2,y_2,z_2), \tag{6}$$
$$\dot{z}_1 = h(x_1,y_1,z_1) \quad \dot{z}_2 = h(x_2,y_2,z_2),$$

here c is coupling strength. We solve simultaneously drive I response systems as given in eq. (6) and for $t \to \infty$, to $|x_1 - x_2|$ and $|y_1 - y_2|$ and $|z_1 - z_2| \to 0$. The equation of stability for the above synchronized system is given by

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = DF.\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \tag{7}$$

ere $(\xi_1,\xi_2,\xi_3) \equiv (x_1 - x_2, y_1 - y_2, z_1 - z_2)$ and DF is acobian $\frac{\partial F_i}{\partial \xi_i}$. The solutions of the eq. (7) convey us eather the synchronization is possible or not. If all solutions re negative then synchronization is stable. As coupling onstant c also occurs in the equation of stability so it ffects the stability of synchronization.

3 *Adaptive control technique :*

n both techniques discussed so far we have considered the lentical synchronization of two identical chaotic systems aving same system parameter. A complicated situation ises when the system parameters and initial conditions are ifferent for both the systems. In such a case synchronization n be achieved by using combination of feedback [12–14] id adaptive control techniques [6,15,16]. We discuss the ethod in brief by considering a nonlinear dynamical system ith three variables x, y and z and a system parameter μ. The olution of the system in terms of flow function f, g and are given as :

$$\dot{x} = f(x,y,z,\mu); \; \dot{y} = g(x,y,z,\mu); \; \dot{z} = h(x,y,z,\mu). \tag{8}$$

this technique we start with two chaotic systems having me functional form but initial conditions and system rameters are different. In response system we apply an ditional dynamics on the system parameter so as to ptively evolve as drive system. The drive and response stems will be

$$= f(x_1,y_1,z_1,\mu_1), \quad \dot{x}_2 = f(x_2,y_2,z_2,\mu_2) - c.(x_1 - x_2),$$
$$= g(x_1,y_1,z_1,\mu_1), \quad \text{and} \quad \dot{y}_2 = g(x_2,y_2,z_2,\mu_2), \tag{9}$$
$$= h(x_1,y_1,z_1,\mu_1) \quad \dot{z}_2 = h(x_2,y_2,z_2,\mu_2),$$
$$\dot{\mu}_2 = -\delta.(\mu_2 - \mu_1)$$

solve simultaneously dirve and response systems given eq. (9), for $t \to \infty$ leads to $|x_1 - x_2|$, $|y_1 - y_2|$, $|z_1 - z_2|$ $|\mu_1 - \mu_2| \to 0$.

The condition for stable synchronization is that the real part of all eigenvalues of Jacobian matrix $DF = \frac{\partial F_i}{\partial \xi_i}$ should be negative, here $(\xi_1, \xi_2, \xi_3, \xi_4) \equiv (x_1 - x_2, y_1 - y_2, z_1 - z_2, \mu_1 - \mu_2)$.

3. Results and discussions

In this section we present the results of our calculation of identical synchronization using complete replacement, feedback and adaptive control techniques in the *Jerk equation* {Eq. (1)}. We are considering the chaotic case corresponding to the control parameter value $A = 0.6$ and we have taken here $\dot{x} = y$ and $\dot{y} = z$.

In Figure 2 we have shown the results using complete replacement of x-variable. It may be seen from Figures 2a and 2b that as t increases the difference between y-variables

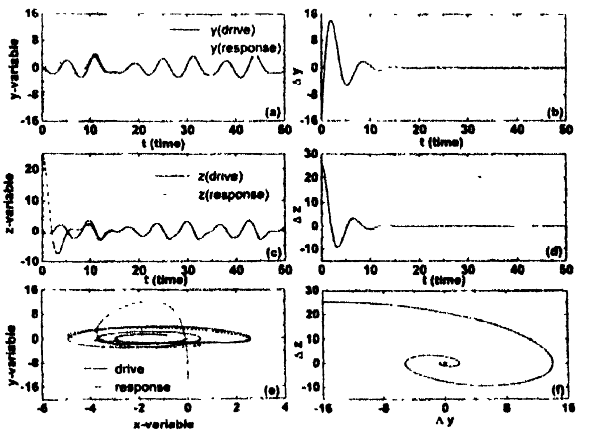

Figure 2. Identical synchronization of chaotic *Jerk dynamical system* {eq.(1)} using complete replacement of x-variable : We show in Frame (a) time history of the y-variables of the drive and response systems and in Frame (b) the difference between y-variables of the drive and response systems as a function of time. In Frames (c) and (d) we depict the same features as in Frames (a) and (b) except that it is for z-variables. Trajectories of the drive and response systems in xy-plane are shown in Frame (e) and in $\Delta x\Delta y$-plane in Frame (f).

of drive and response systems vanishes. Similar behaviour is discussed in Figures 2c and 2d for z-variables. In Figure 2e we have shown the trajectories of the drive and response systems in the *xy-plane*. We see that the drive and response start with different initial conditions but after some time both converge to the same trajectory. A better view has been shown in Figure 2f in which we have plotted the trajectory in the *ΔyΔz-plane* (*i.e.* difference plane) this is a spiral ending at a fixed point (0,0).

When we use the complete replacement of y and z variables the synchronization is not possible because of stability conditions. In Table 3·1 we have shown the conditional Lyapunov exponents (CLE's) for different combination of drive variables and response subsystems in *Jerk equation*. It is clear from the Table 3.1 that synchronization is only possible with complete replacement of x-variable.

Table 1. Conditional Lyapunov Exponents (CLE's) for different combinations of drive variables and response subsystems.

$\dddot{x} + A\ddot{x} + x = G(x)$; $G(x) = |x| - 2$, $A = 0.6$

Drive variable	Response subsystem	Conditional Lyapunov exponents
x	yz	-0.3, -0.3
y	xz	0.8×10^{-6}, 0.0
z	xy	0.0, 0.0

In Figure 3 we have shown the results of identical synchronization using feedback method and x as a drive variable where coupling strength $c = 10$. In Figure 3a we have shown the time history of x-variables of drive and response systems and in Figure 3b the difference in x-variables of drive and response systems as a function of time. Similarly Figures 3c and 3d for y-variables and Figures 3e and 3f for z-variables. We predict that synchronization is only possible with $c > 0.799$. By using y and z as a drive variable for feedback, synchronization will not be stable as the CLE's become positive for these combinations.

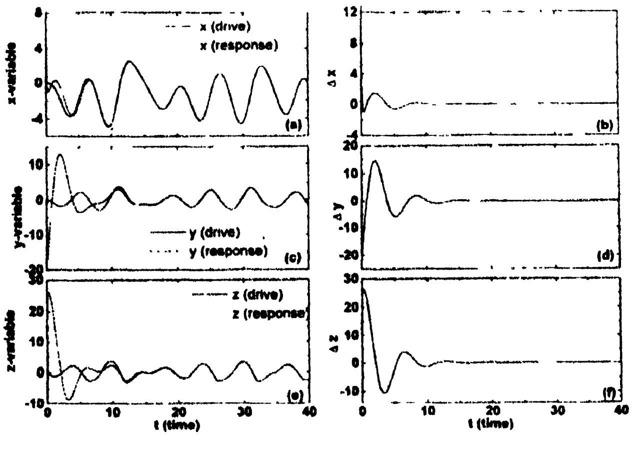

Figure 3. Identical synchronization of chaotic *Jerk dynamical system* {eq. (1)} using feedback technique (coupling strength $c = 10$) with x as a drive variable. We show in Frame (a) time history of the X-variables of the drive and response systems and in Frame (b) the difference between x-variables of the drive and response systems as a function of time. In Frames (c) and (d) we depict the same features as in Frames (a) and (b) except that it is for y-variables. In Frames (e) and (f) we have depicted the same features as in Frames (a) and (b) except that it is for z-variables.

In Figure 4 we have presented the result of identical synchronization using complete replacement (Figures 4a and 4b) and feed back technique (Figures 4c and 4d) when the parameter for the drive and response systems differ by 5% *i.e.*, $A_d = 0.6$ and $A_r = 0.63$. We observe that the system is partially synchronized and response variables remain within neighborhood of the drive variables. In Figure 5 we have presented the result of the non-identical synchronization using adaptive control algorithm and x as a drive variable.

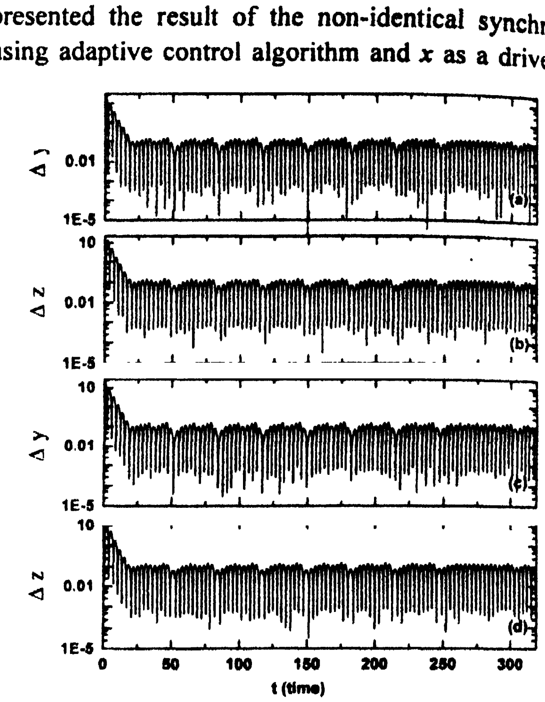

Figure 4. Identical synchronization of chaotic *Jerk dynamical system* {eq. (1)} when system parameter for drive and response systems differ by 5% *i.e.* $A_d = 0.6$ and $A_r = 0.63$. Frames (a) and (b) show the absolute value of the differences between y and z-variables of drive and response systems by using complete replacement of x-variable. Frames (c) and (d) are respectively same as Frames (a) and (b) except for use of feedback technique and x as a drive variable.

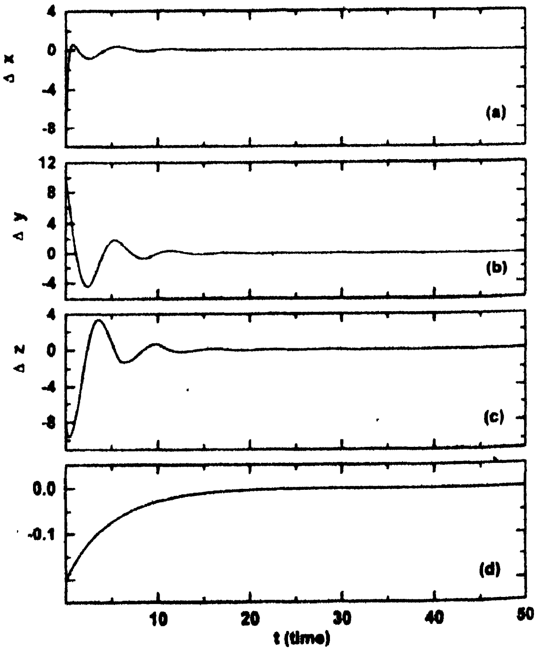

Figure 5. Non-identical synchronization of chaotic *Jerk dynamical system* {eq. (1)} when drive and response systems have different value of system parameter by using adaptive control algorithm and x as a drive variable ($c = 5$ and $\delta = 0.02$) : Frame (a) shows the difference between x-variables of drive and response systems as a function of time. Frames (b), (c) and (d) are same as Frame (a) except that it is for y-variable, z-variable and system parameter respectively.

.= 5 and $\delta = 0.02$). We predict that for only certain range f c and δ values synchronization is possible.

In the present analysis we have discussed numerically e synchronization of chaotic *Jerk dynamical systems* using rive response, feedback and adaptive control techniques. rk *dynamical systems* are very simple and it is very easy construct circuit representation of these systems using sistors, diodes and operational amplifiers. These circuits oduce chaotic signals for a certain range of control rameter. This property makes these systems useful for plication of communication purpose as an information-aring message can be hidden within complicated structure the chaotic carrier. This chaotic carrier signal may be constructed at the receiver using synchronization chniques and subtracted from the transmitted signal (hidden essage + chaotic carrier). Therefore we will be left with ily our hidden signal. In this analysis we have monstrated synchronization of the *Jerk dynamical system* using simple techniques. However, it is very easy to troduce noise for such type of circuits. Hence nchronization of these systems subjected to common noise ill also be an important issue for communication. We have ken in the present study the same form of $G(x)$ for both e transmitter and receiver. However, there is a possibility synchronization when the transmitter and receiver have fferent forms of $G(x)$. The investigation using different rms of $G(x)$ is in progress and will be reported elsewhere.

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